# Solutions to tutorial exercises for stochastic processes

T1. Let $G(x,y)$ be the Green function of the Markov chain. Let $x$ be the recurrent state, so that $G(x,x)=\infty$. We will show that state $y$ is recurrent by showing that $G(y,y)=\infty$. Let $t,s\geq 0$. We use the Chapman-Kolmogorov equation twice to obtain

$$p_{2t+s}(y,y)\geq p_t(y,x)p_{t+s}(x,y)\geq p_t(y,x)p_s(x,x)p_t(x,y).$$

Therefore,

$$G(y,y)=\int_0^\infty p_s(y,y)\mathrm{d}s\geq\int_{2t}^\infty p_s(y,y)\mathrm{d}s=\int_0^\infty p_{2t+s}(y,y)\mathrm{d}s\geq p_t(y,x)G(x,x)p_t(x,y).$$

Since the Markov chain is irreducible we have $p_t(y,x),p_t(x,y)>0$, we conclude that $G(y,y)=\infty$.

T2. '$\Rightarrow$': Using strict stationarity of the Markov chain we find

$$\pi(x)=\mathbb{P}(X_0=x)=\mathbb{P}(X_t=x)=\sum_{x_0\in S}\mathbb{P}(X_t=x,X_0=x_0)=\sum_{x_0\in S}\pi(x_0)p_t(x_0,x),$$

so $\pi(\cdot)$ is invariant.

'$\Leftarrow$': Let $n\in\mathbb{N}$, $0\leq t_1<t_2<\cdots<t_n$ and $s>0$. Let $x_1,\dots,x_n\in S$. We use the invariance property of $\pi(\cdot)$ to obtain

$$\begin{aligned}\mathbb{P}(X_{t_1+s}=x_1,\dots,X_{t_n+s}=x_n)&=\sum_{x_0\in S}\pi(x_0)p_{t_1+s}(x_0,x_1)p_{t_2-t_1}(x_1,x_2)\dots p_{t_n-t_{n-1}}(x_{n-1},x_n)\\&=\pi(x_1)p_{t_2-t_1}(x_1,x_2)\dots p_{t_n-t_{n-1}}(x_{n-1},x_n),\end{aligned}$$

which is independent of $s$, so the Markov chain is strictly stationary.

T3. Let $f$ be a non-negative harmonic function for $X$. Then

$$|f(x)|=f(x)=\mathbb{E}[f(x)]=\mathbb{E}\big[|f(x)|\big]<\infty,$$

so that $f$ is bounded. Since $X$ is irreducible and recurrent it follows that every bounded harmonic function for $X$ is constant.

T4. '⇒': By reversibility we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\pi(x)p_t(x,y)\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\pi(y)p_t(y,x)\Big|_{t=0} \qquad \forall x,y \in S \\
\pi(x)q(x,y) = \pi(y)q(y,x)& \qquad\qquad\qquad \forall x,y \in S.
\end{aligned}
$$

'⇐': Assume that $\pi(x)q(x,y) = \pi(y)q(y,x)$ for all $x,y \in S$. This can be stated as

$$
\begin{pmatrix} \pi(1) & & \\ & \ddots & \\ & & \pi(n) \end{pmatrix} Q = Q^T \begin{pmatrix} \pi(1) & & \\ & \ddots & \\ & & \pi(n) \end{pmatrix}.
$$

Since $S$ is finite the transition probabilities $P_t$ are given by $\exp(tQ)$. Therefore

$$
\begin{aligned}
\begin{pmatrix} \pi(1) & & \\ & \ddots & \\ & & \pi(n) \end{pmatrix} P_t &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix} \pi(1) & & \\ & \ddots & \\ & & \pi(n) \end{pmatrix} Q^k \\
&= \begin{pmatrix} \pi(1) & & \\ & \ddots & \\ & & \pi(n) \end{pmatrix} + \sum_{k=1}^{\infty} \frac{t^k}{k!} Q^T \begin{pmatrix} \pi(1) & & \\ & \ddots & \\ & & \pi(n) \end{pmatrix} Q^{k-1} \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(Q^T\right)^k \begin{pmatrix} \pi(1) & & \\ & \ddots & \\ & & \pi(n) \end{pmatrix} \\
&= e^{tQ^T} \begin{pmatrix} \pi(1) & & \\ & \ddots & \\ & & \pi(n) \end{pmatrix} \\
&= P_t^T \begin{pmatrix} \pi(1) & & \\ & \ddots & \\ & & \pi(n) \end{pmatrix}.
\end{aligned}
$$

It follows that $\pi(\cdot)$ is reversible.